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New systems of generalized vector quasi-equilibrium problems in product FC-spaces

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Abstract The notions of $C_i(x)$-FC-diagonally quasiconvex, $C_i(x)$-FC-quasiconvex and $C_i(x)$-FC-quasiconvex-like for set-valued mappings are introduced in FC-spaces without convexity structure. By applying these notions and a maximal element theorem for a family of set-valued mappings on product FC-space due to author, some new existence theorems of solutions for four new classes of systems of generalized vector quasi-equilibrium problems are proved in noncompact FC-spaces. These results improve and generalize some recent known results in literature to noncompact FC-spaces.

Keywords Maximal element · System of generalized vector quasi-equilibrium problems · FC-hull · $C_i(x)$-FC-diagonally quasiconvex · $C_i(x)$-FC-quasiconvex · $C_i(x)$-FC-quasiconvex-like · FC-spaces

1 Introduction

Let X and Y be two nonempty sets. We denote by 2^Y and $< X >$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. Let I be any index set. For each $i \in I$, let X_i and Y_i be topological spaces and Z_i be a nonempty set. Let $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$ and for $i \in I$ and $x \in X$, $x_i = \pi_i(x)$ be the projection of x onto X_i. For each $i \in I$, let $A_i : X \times Y \to 2^{X_i}$, $T_i : X \times Y \to 2^{Y_i}$, $C_i : X \to 2^{Z_i}$, and $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ be set-valued mappings.

In this paper, we consider the following systems of generalized vector quasi-equilibrium problems:

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(I) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \not\subset C_i(\hat{x}), \quad \forall\ z_i \in A_i(\hat{x}, \hat{y}). \tag{SGVQEP(I)}$$

(II) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \subset C_i(\hat{x}), \quad \forall\ z_i \in A_i(\hat{x}, \hat{y}). \tag{SGVQEP(II)}$$

(III) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \bigcap C_i(\hat{x}) = \emptyset, \quad \forall\ z_i \in A_i(\hat{x}, \hat{y}). \tag{SGVQEP(III)}$$

(IV) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \bigcap C_i(\hat{x}) \neq \emptyset, \quad \forall\ z_i \in A_i(\hat{x}, \hat{y}). \tag{SGVQEP(IV)}$$

The SGVQEP (I)–SGVQEP (IV) were introduced and studied by Ding [1] in locally FC-uniform spaces. The SGVQEP (I)–SGVQEP (IV) with $A_i(x, y) = A_i(x)$ and $T_i(x, y) = T_i(x)$ for all $i \in I$ and $(x, y) \in X \times Y$ were introduced and studied by Lin and Liu [2], Lin et al. [3], Lin [4], Peng et al. [5], Ding [6,7] and Ding and Yao [8] in the cone setting of (locally convex) topological vector spaces, FC-spaces, locally FC-uniform spaces and G-convex spaces, respectively. The SGVQEP (I)–SGVQEP (IV) with $Y_i = X_i$, $A_i(x, y) = A_i(x)$, $T_i(x, y) = T_i(x)$ and $\psi_i : X \times Y_i \times X_i \to 2^{Z_i}$ for each $i \in I$ and $(x, y) \in X \times Y$ were introduced and studied by Lin [9], Ding et al. [10], and Ding [11] in topological vector spaces, locally G-convex uniform spaces and locally FC-spaces, respectively. Some existence theorems of solutions for SGVQEP (I)–SGVQEP (IV) were established under different assumptions.

For appropriate choices of the index set I, the spaces X_i, Y_i, Z_i and the mappings A_i, T_i, C_i and ψ_i, it is easy to see that the SGVQEP (I)–SGVQEP (VI) include most extensions and generalizations of the systems of generalized (vector) quasi-equilibrium problems, the systems of generalized (vector) quasi-variational inequality problems, the systems of generalized (vector) equilibrium problems and the systems of generalized (vector) variational inequality problems as very special cases, for example, see [1–11] and the references therein.

In this paper, we introduce the new notions of $C_i(x)$-FC-diagonally quasiconvex, $C_i(x)$-FC-quasiconvex and $C_i(x)$-FC-quasiconvex-like for set-valued mappings in FC-space. By using these notions and an existence theorem of maximal elements for a family of set-valued mappings due to author [12], some new existence theorems of solutions for the SGVQEP (I)–SGVQEP (IV) are proved in noncompact FC-spaces without convexity structure. These results improve and generalize some recent known results in literature to noncompact FC-spaces.

2 Preliminaries

Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \cdots, e_n$. If J is a nonempty subset of $\{0, 1, \cdots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$.

The following notion was introduced by Ben-El-Mechaiekh et al. [13].

Definition 2.1 (X, Γ) is said to be a L-convex space if X is a topological space and $\Gamma : < X > \to 2^X$ is a mapping such that for each $N \in < X >$ with $|N| = n + 1$, there exists a continuous mapping $\varphi_N : \Delta_n \to \Gamma(N)$ satisfying $A \in < N >$ with $|A| = J + 1$ implies $\varphi_N(\Delta_J) \subset \Gamma(A)$, where Δ_J is the face of Δ_N corresponding to A.

The following notion of a finitely continuous topological space (in short, FC-space) was introduced by Ding [14].

Definition 2.2 (X, φ_N) is said to be a FC-space if X is a topological space and for each $N = \{x_0, \cdots, x_n\} \in < X >$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. A subset D of (X, φ_N) is said to be a FC-subspace of X if for each $N = \{x_0, \cdots, x_n\} \in < X >$ and for each $\{x_{i_0}, \cdots, x_{i_k}\} \subset N \bigcap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \cdots, k\})$.

It is clear that each L-convex space must be a FC-space. The following example shows that there exists a FC-space which is not a L-convex space.

Example 2.1 Let $X = (1, 2) \bigcup (3, +\infty)$ with usual topology. Define a mapping φ_N: $< X >: \Delta_n \to 2^X$ as follows: for each $N = \{x_0, \cdots, x_n\} \in < X >$,

$$\varphi_N(\alpha) = \begin{cases} \sum_{i=0}^n \alpha_i x_i, & \text{if } N \subset (1, 2), \\ 3\sum_{i=0}^n \alpha_i x_i, & \text{if } N \not\subset (1, 2), \ \forall\, \alpha = (\alpha_0, \cdots, \alpha_n) \in \Delta_n. \end{cases}$$

It is easy to see that φ_N is continuous and hence (X, φ_N) is a FC-space. For any $a \in (1, 2)$ and $b \in (3, \infty)$, $(a, 2) \bigcup (3, \infty)$ and (b, ∞) are both FC-subspace of X. But X is not convex.

If we define a set-valued mapping $\Gamma :< X > \to 2^X$ by

$$\Gamma(N) = \varphi_N(\Delta_n), \quad \forall\, N = \{x_0, \cdots, x_n\} \in < X >,$$

then we have that for each $N = \{x_0, \cdots, x_n\} \in < X >$, $\varphi_N(\Delta_N) \subset \Gamma(N)$. But if $N = N_1 \bigcup N_2$ where $N_1 \subset (1, 2)$ with $|N_1| = J + 1$, $J < n$ and $N_2 \subset (3, +\infty)$, then we have $\Gamma(N_1) = \varphi_{N_1}(\Delta_J) \subset (1, 2)$ and $\varphi_N(\Delta_J) \subset (3, +\infty)$, i.e., $\varphi_N(\Delta_J) \not\subset \Gamma(N_1)$. Hence (X, Γ) is not a L-convex space.

It is clear that any convex subset of a topological vector space, any H-space introduced by Horvath [15], any G-convex space introduced by Park and Kim [16] and any L-convex spaces introduced by Ben-El-Mechaiekh et al. [13] are all FC-space. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces.

By the definition of FC-subspaces of a FC-space, it is easy to see that if $\{B_i\}_{i\in I}$ is a family of FC-subspaces of a FC-space (Y, φ_N) and $\bigcap_{i\in I} B_i \neq \emptyset$, then $\bigcap_{i\in I} B_i$ is also a FC-subspace of (Y, φ_N) where I is any index set. For a subset A of (Y, φ_N), we can define the FC-hull of A as follows:

$$FC(A) = \bigcap\{B \subset Y : A \subset B \text{ and } B \text{ is } FC - \text{subspace of } Y\}.$$

Clearly, $FC(A)$ is the smallest FC-subspace of Y containing A and each FC-subspace of a FC-space is also a FC-space

Lemma 2.1 [12] *Let (Y, φ_N) be a FC-space and A be a nonempty subset of Y. Then*

$$FC(A) = \bigcup\{FC(N) : N \in < A >\}.$$

Remark 2.1 Lemma 2.1 generalizes Lemma 1 of Tarafdar [17] and Lemma 2.1 of Tan and Zhang [18] from H-space and G-convex space to FC-space without any convexity structure.

Lemma 2.2 [12] *Let X be a topological space, (Y, φ_N) be a FC-space and $G : X \to 2^Y$ be such that $G^{-1}(y) = \{x \in X : y \in G(x)\}$ is compactly open in X for each $y \in Y$. Then the mapping $FC(G) : X \to 2^Y$ defined by $FC(G)(x) = FC(G(x))$ for each $x \in X$ satisfies that $(FC(G))^{-1}(y)$ is also compactly open in X for each $y \in Y$.*

Remark 2.2 Lemma 2.2 generalizes Lemma 3.1 of Ding [19] and Lemma 2.2 of Tan and Zhang [18] from H-space and G-convex spaces to FC-space without convexity structure.

Lemma 2.3 [14] *Let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be a FC-space. Let $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then (Y, φ_N) is also a FC-space.*

The following result is a special case of Corollary 3.3 of Ding [12].

Lemma 2.4 *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a FC-space, $X = \prod_{i\in I} X_i$ and K be a compact subset of X. For each $i \in I$, let $G_i : X \to 2^{X_i}$ be such that*

- (i) *for each $i \in I$ and $x \in X$, $G_i(x)$ is a FC-subspace of X_i,*
- (ii) *for each $x \in X$, $\pi_i(x) \notin G_i(x)$ for all $i \in I$,*
- (iii) *for each $y_i \in X_i$, $G_i^{-1}(y_i)$ is compactly open in X*
- (iv) *for each $N_i \in< X_i >$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap G_i(x) \neq \emptyset$. Then there exists $\hat{x} \in K$ such that $G_i(\hat{x}) = \emptyset$, for each $i \in I$.*

3 Existence of solutions for the SGVQEP

Throughout this section, unless otherwise specified, we shall fix the following notations and assumptions. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ'_{N_i}) be FC-spaces, and Z_i be a nonempty set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : X \times Y \to 2^{X_i}$, $T_i : X \times Y \to 2^{Y_i}$, $C_i : X \to 2^{Z_i}$ and $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ be set-valued mappings.

Definition 3.1 For each $i \in I$ and $y \in Y$, ψ_i is said to be

- (i) $C_i(x)$-FC-diagonally quasiconvex of type (I) in third argument if each $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in< X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(x, y, z_{i,j}) \not\subset C_i(x)$,
- (ii) $C_i(x)$-FC-diagonally quasiconvex of type (II) in third argument if for each $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in< X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(x, y, z_{i,j}) \subset C_i(x)$,
- (iii) $C_i(x)$-FC-diagonally quasiconvex of type (III) in third argument if for each $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in< X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(x, y, z_{i,j}) \bigcap C_i(x) = \emptyset$,
- (iv) $C_i(x)$-FC-diagonally quasiconvex of type (IV) in third argument if for each $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in< X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(x, y, z_{i,j}) \bigcap C_i(x) \neq \emptyset$.

Remark 3.1 The notions in Definition 3.1 generalizes the corresponding notions of Peng et al. [5] from convex subsets of topological vector spaces to FC-spaces.

Lemma 3.1 [20] *Let X and Y be topological spaces and $G : X \to 2^Y$ be a set-valued mapping. Then G is lower semicontinuous in $x \in X$ if and only if for any $y \in G(x)$ and any net $\{x_\alpha\} \subset X$ satisfying $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in G(x_\alpha)$ and $y_\alpha \to y$.*

The following results are Proposition 4.5 and proposition 4.6 of Ding [21].

Lemma 3.2 *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

(i) *C has closed (resp., open) graph,*
(ii) *for each $y \in Y$, $F(\cdot, y)$ is lower semicontinuous on each compact subset of X.*

Then the mapping $F^* : Y \to 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \bigcap C(x) = \emptyset\}$) has compactly closed values.

Lemma 3.3 *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

(i) *C has open (resp., closed) graph in $X \times Z$,*
(ii) *for each $y \in Y$, $F(\cdot, y)$ is upper semicontinuous on each compact subset of X with nonempty compactly closed values.*

Then the mapping $F^* : Y \to 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \not\subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \bigcap C(x) \neq \emptyset\}$) has compactly closed values.

Remark 3.2 Lemma 3.3 generalizes Lemma 2.3 of Ding and Park [22].

Theorem 3.1 *Suppose that K and H are nonempty compact subsets of X and Y, respectively, such that for each $i \in I$,*

(i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $T_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, respectively,*
(ii) *for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ and the set $\{(x, y) \in X \times Y : \psi_i(x, y, u_i) \subset C_i(x)\}$ are all compactly open in $X \times Y$,*
(iii) *for each $y \in Y$, ψ_i is $C_i(x)$-FC-diagonally quasiconvex of type (I) in third argument,*
(iv) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y), \ y_i \in T_i(x, y)\}$ is compactly closed in $X \times Y$,*
(v) *for each $N_i \in \langle X_i \rangle$, there exists compact FC-subspace L_{N_i} containing N_i and for each $M_i \in \langle Y_i \rangle$, there exists compact FC-subspace L_{M_i} of Y_i containing M_i,*
(vi) *for each $(x, y) \in X \times Y \setminus K \times H$, there exist $i \in I$, $\bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ and $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ such that $\psi_i(x, y, \bar{z}_i) \subset C_i(x)$.*

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \ \hat{y}_i \in T_i(\hat{x}, \hat{y}) \text{ and } \psi_i(\hat{x}, \hat{y}, z_i) \not\subset C_i(\hat{x}), \quad \forall \ z_i \in A_i(\hat{x}, \hat{y})$$

i.e., $(\hat{x}, \hat{y})$ is a solution of the SGVQEP(I).

Proof For each $i \in I$, define a set-valued mapping $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : \psi_i(x, y, z_i) \subset C_i(x)\}, \quad \forall \ (x, y) \in X \times Y.$$

We show that for each $i \in I$ and $(x, y) \in X \times Y$,

$$x_i = \pi_i(x) \notin FC(P_i(x, y)) \tag{3.1}$$

If it is false, then there exist $i \in I$ and $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x}_i = \pi_i(\bar{x}) \in FC(P_i(\bar{x}, \bar{y}))$. By Lemma 2.1, there exists $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in \langle P_i(\bar{x}, \bar{y}) \rangle$ such that $\bar{x}_i = \pi_i(\bar{x}) \in FC(N_i)$. Hence we have

$$\psi_i(\bar{x}, \bar{y}, z_{i,j}) \subset C_i(\bar{x}), \quad \forall \ j = 0, \cdots, n.$$

But, by (iii) and Definition 3.1, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(\bar{x}, \bar{y}, z_{i,j}) \not\subset C_i(\bar{x})$ which is a contradiction. Hence (3.1) is true. By the condition (ii), for each $i \in I$ and $z_i \in X_i$,

$$P_i^{-1}(z_i) = \{(x, y) \in X \times Y : \psi_i(x, y, z_i) \subset C_i(x)\}$$

is compactly open in $X \times Y$. It follows from Lemma 2.2 that $(FC(P_i))^{-1}(z_i)$ is also compactly open in $X \times Y$ for each $z_i \in X_i$. By Lemma 2.3, for each $i \in I$, $X_i \times Y_i$ is a FC-space and $X \times Y$ is also a FC-space. For each $i \in I$, define a set-valued mapping $G_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$G_i(x, y) = \begin{cases} [A_i(x, y) \bigcap FC(P_i(x, y))] \times T_i(x, y), & \text{if } (x, y) \in W_i, \\ A_i(x, y) \times T_i(x, y), & \text{if } (x, y) \notin W_i, \end{cases}$$

By the condition (i), for each $i \in I$ and $(x, y) \in X \times Y$, $G_i(x, y)$ is a FC-subspace of $X_i \times Y_i$. By the definition of W_i and (3.1), for each $i \in I$ and $(x, y) \in X \times Y$, $(x_i, y_i) \notin G_i(x, y)$. For each $i \in I$ and $(u_i, v_i) \in X_i \times Y_i$, we have

$$G_i^{-1}(u_i, v_i) = [A_i^{-1}(u_i) \bigcap (FC(P_i))^{-1}(u_i) \bigcap T_i^{-1}(v_i)] \\ \bigcup [(X \times Y \setminus W_i) \bigcap A_i^{-1}(u_i) \bigcap T_i^{-1}(v_i)].$$

Since $(FC(P_i))^{-1}(u_i)$ is compactly open in $X \times Y$ for each $u_i \in X_i$, by the condition (ii), $G_i^{-1}(u_i, v_i)$ is also compactly open in $X \times Y$. By (v), for each $H_i = N_i \times M_i \in < X_i \times Y_i >$ there exists compact FC-subspace $L_{H_i} = L_{N_i} \times L_{M_i}$ of $X_i \times Y_i$ containing H_i. By (vi), for each $(x, y) \in X \times Y \setminus K \times H$, there exists $i \in I$ such that $G_i(x, y) \bigcap L_{H_i} \neq \emptyset$. All conditions of Lemma 2.4 are satisfied. By Lemma 2.4, there exists $(\hat{x}, \hat{y}) \in K \times H$ such that $G_i(\hat{x}, \hat{y}) = \emptyset$ for each $i \in I$. If $(\hat{x}, \hat{y}) \notin W_j$ for some $j \in I$, then either $A_j(\hat{x}, \hat{y}) = \emptyset$ or $T_j(\hat{x}, \hat{y}) = \emptyset$ which contradicts the condition (i). Therefore $(\hat{x}, \hat{y}) \in W_i$ for each $i \in I$. This shows that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in T_i(\hat{x}, \hat{y})$ and $A_i(\hat{x}, \hat{y}) \bigcap FC(P_i(\hat{x}, \hat{y})) = \emptyset$ and hence $A_i(\hat{x}, \hat{y}) \bigcap P_i(\hat{x}, \hat{y}) = \emptyset$. Therefore, for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \not\subset C_i(\hat{x}), \quad \forall\, z_i \in A_i(\hat{x}, \hat{y}),$$

i.e., $(\hat{x}, \hat{y})$ is a solution of the SGVQEP(I).

Remark 3.3 Theorem 3.1 generalizes Theorem 3.1 of Peng et al. [5] in the following ways: (1) the mathematical model in Theorem 3.1 is more general than that in [5]; (2) from nonempty convex subsets of topological vector spaces to FC-spaces without convexity structure; (3) for each $i \in I$, Z_i may be any nonempty set and $C_i(x)$ may not have cone structure; (4) for each $i \in I$ and $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ may be compactly open in $X \times Y$ and the set W_i may be compactly closed in $X \times Y$. Theorem 3.1 also generalizes Theorems 4.1 and 4.2 of Ding [6], Theorem 4.1 of Ding and Yao [8] in several aspects.

Remark 3.4 The condition (iii) of Theorem 3.1 can be replaced by the following conditions:
(iii)$_1$ for each $(x, y) \in X \times Y$, the set $P_i(x, y) = \{z_i \in X_i : \psi_i(x, y, z_i) \subset C_i(x)\}$ is a FC-subspace of X_i,
(iii)$_2$ for all $(x, y) \in X \times Y$, $\psi_i(x, y, x_i) \not\subset C_i(x)$.

In fact, if the condition (iii) of Theorem 3.1 does not hold, then there exist $i \in I$, $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >$ and $\bar{x} \in X$ with $\bar{x}_i \in FC(N_i)$ such that for all $j = 0, \cdots n$, ψ_i $(\bar{x}, y, z_{i,j}) \subset C_i(\bar{x})$. It follows that $N_i \subset P_i(\bar{x}, y)$. Since $P_i(\bar{x}, y)$ is a FC-subspace by (iii)$_1$, we have $\bar{x}_i \in FC(N_i) \subset P_i(\bar{x}, y)$ and hence $\psi_i(\bar{x}, y, \bar{x}_i) \subset C_i(\bar{x})$ which contradicts the condition (iii)$_2$. Therefore the condition (iii) of Theorem 3.1 must be hold.

Remark 3.5 If for each $i \in I$, Z_i is a topological space. The condition that for each $z_i \in X_i$, the set $\{(x, y) \in X \times Y : \psi_i(x, y, z_i) \subset C_i(x)\}$ is compactly open in $X \times Y$ in the condition (ii) of Theorem 3.1 can be replaced by the following conditions:
(ii)$_1$ $C_i : X \to 2^{Z_i}$ has open graph in $X \times Z_i$,
(ii)$_2$ for each $z_i \in X_i$, the mapping $(x, y) \mapsto \psi_i(x, y, z_i)$ is upper semicontinuous on each compact subsets of $X \times Y$ with compact values.

In fact, from the conditions (ii)$_1$, (ii)$_2$ and Lemma 3.3 it follows that for each $z_i \in X_i$, the set $\{(x, y) \in X \times Y : \psi_i(x, y, z_i) \not\subset C_i(x)\}$ is compactly closed in $X \times Y$. Hence the set $\{(x, y) \in X \times Y : \psi_i(x, y, z_i) \subset C_i(x)\}$ is compactly open in $X \times Y$.

By using similar argument as in the proof of Theorem 3.1, we can prove the following result.

Theorem 3.2 *Suppose that K and H are nonempty compact subsets of X and Y, respectively, such that for each $i \in I$,*

(i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $T_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, respectively,*
(ii) *for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ and the set $\{(x, y) \in X \times Y : \psi_i(x, y, u_i) \not\subset C_i(x)\}$ are all compactly open in $X \times Y$,*
(iii) *for each $y \in Y$, ψ_i is $C_i(x)$-FC-diagonally quasiconvex of type (II) in third argument,*
(iv) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in T_i(x, y)\}$ is compactly closed in $X \times Y$,*
(v) *for each $N_i \in < X_i >$, there exists compact FC-subspace L_{N_i} containing N_i and for each $M_i \in < Y_i >$, there exists compact FC-subspace L_{M_i} of Y_i containing M_i,*
(vi) *for each $(x, y) \in X \times Y \setminus K \times H$, there exist $i \in I$, $\bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ and $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ such that $\psi_i(x, y, \bar{z}_i) \not\subset C_i(x)$.*

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \subset C_i(\hat{x}), \quad \forall\ z_i \in A_i(\hat{x}, \hat{y})$$

i.e., $(\hat{x}, \hat{y})$ is a solution of the SGVQEP(II).

Remark 3.6 Theorem 3.2 generalizes Theorem 4.3 of Ding [6] to more general mathematical model. Theorem 3.2 also generalizes Theorem 3.3 of Peng et al. [5] from convex subsets of topological vector spaces to FC-spaces without convexity structure under much weak assumptions.

Theorem 3.3 *Suppose that K and H are nonempty compact subsets of X and Y, respectively, such that for each $i \in I$,*

(i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $T_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, respectively,*
(ii) *for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ and the set $\{(x, y) \in X \times Y : \psi_i(x, y, u_i) \bigcap C_i(x) \neq \emptyset\}$ are all compactly open in $X \times Y$,*
(iii) *for each $y \in Y$, ψ_i is $C_i(x)$-FC-diagonally quasiconvex of type (III) in third argument,*
(iv) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in T_i(x, y)\}$ is compactly closed in $X \times Y$,*
(v) *for each $N_i \in < X_i >$, there exists compact FC-subspace L_{N_i} containing N_i and for each $M_i \in < Y_i >$, there exists compact FC-subspace L_{M_i} of Y_i containing M_i,*
(vi) *for each $(x, y) \in X \times Y \setminus K \times H$, there exist $i \in I$, $\bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ and $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ such that $\psi_i(x, y, \bar{z}_i) \bigcap C_i(x) \neq \emptyset$.*

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \bigcap C_i(\hat{x}) = \emptyset, \quad \forall\ z_i \in A_i(\hat{x}, \hat{y})$$

i.e., $(\hat{x}, \hat{y})$ *is a solution of the SGVQEP(III).*

Proof For each $i \in I$, define a set-valued mapping $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : \psi_i(x, y, z_i) \bigcap C_i(x) \neq \emptyset\}, \quad \forall\, (x, y) \in X \times Y.$$

We show that for each $i \in I$ and $(x, y) \in X \times Y$,

$$x_i = \pi_i(x) \notin FC(P(x, y)). \tag{3.2}$$

If it is false, then there exist $i \in I$ and $(\bar{x}, \bar{y}) \in X \times Y$ such that $x_i = \pi_i(\bar{x}) \in FC(P(\bar{x}, \bar{y}))$. By Lemma 2.1, there exists $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in< P_i(\bar{x}, \bar{y}) >$ such that $\bar{x}_i \in FC(N_i)$. Hence we have

$$\psi_i(\bar{x}, \bar{y}, z_{i,j}) \bigcap C_i(\bar{x}) \neq \emptyset, \quad \forall\ j = 0, \cdots, n.$$

But, by (iii) and Lemma 3.1, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(\bar{x}, \bar{y}, z_{i,j}) \bigcap C_i(\bar{x}) = \emptyset$ which is a contradiction. Hence (3.2) is true. By the condition (ii), for each $i \in I$ and $z_i \in X_i$,

$$P_i^{-1}(z_i) = \{(x, y) \in X \times Y : \psi_i(x, y, z_i) \bigcap C_i(x) \neq \emptyset\}$$

is compactly open in $X \times Y$. It follows from Lemma 2.2 that $(FC(P_i))^{-1}(z_i)$ is also compactly open in $X \times Y$ for each $z_i \in X_i$. By Lemma 2.3, for each $i \in I$, $X_i \times Y_i$ is a FC-space and $X \times Y$ is also a FC-space. The rest of the proof is similar to that of Theorem 3.1. We omit.

Remark 3.7 Theorem 3.3 generalizes Theorem 3.2 of Peng et al. [5] in the following ways: (1) from nonempty convex subsets of topological vector spaces to FC-spaces without any convexity structure; (2) for each $i \in I$, Z_i may be any nonempty set and $C_i(x)$ may not have cone structure; (3) for each $i \in I$ and $(u_i, v_i) \in\in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ may be compactly in $X \times Y$ and the set W_i may be compactly closed in $X \times Y$.

Theorem 3.4 *Suppose that* K *and* H *are nonempty compact subsets of* X *and* Y, *respectively, such that for each* $i \in I$,

- (i) *for each* $(x, y) \in X \times Y$, $A_i(x)$ *and* $T_i(x)$ *are both nonempty* FC*-subspaces of* X_i *and* Y_i, *respectively,*
- (ii) *for each* $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ *and the set* $\{(x, y) \in X \times Y : \psi_i(x, y, u_i) \bigcap C_i(x) = \emptyset\}$ *are all compactly open in* $X \times Y$,
- (iii) *for each* $y \in Y$, ψ_i *is* $C_i(x)$*-FC-diagonally quasiconvex of type (IV) in third argument,*
- (iv) *the set* $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x),\ y_i \in T_i(x)\}$ *is compactly closed in* $X \times Y$,
- (v) *for each* $N_i \in< X_i >$, *there exists compact* FC*-subspace* L_{N_i} *containing* N_i *and for each* $M_i \in< Y_i >$, *there exists compact* FC*-subspace* L_{M_i} *of* Y_i *containing* M_i,
- (vi) *for each* $(x, y) \in X \times Y \setminus K \times H$, *there exist* $i \in I$, $\bar{z}_i \in A_i(x) \bigcap L_{N_i}$ *and* $\bar{y}_i \in T_i(x) \bigcap L_{M_i}$ *such that* $\psi_i(x, y, \bar{z}_i) \bigcap C_i(x) = \emptyset$.

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \bigcap C_i(\hat{x}) \neq \emptyset, \quad \forall\ z_i \in A_i(\hat{x})$$

i.e., $(\hat{x}, \hat{y})$ *is a solution of the SGVQEP(IV).*

Proof For each $i \in I$, define a set-valued mapping $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : \psi_i(x, y, z_i) \bigcap C_i(x) = \emptyset\}, quad \forall\ (x, y) \in X \times Y.$$

By using similar argument as in the proof of Theorem 3.3, we can easily prove that the conclusion of Theorem 3.4 holds.

Remark 3.8 Theorem 3.4 generalizes Theorem 3.4 of Peng et al. [6] in the following ways: (1) the mathematical model in Theorem 3.1 is more general than that in [5]; (2) from nonempty convex subsets of topological vector spaces to FC-spaces without convexity structure; (3) for each $i \in I$, Z_i may be any nonempty set and $C_i(x)$ may not have cone structure; (4) for each $i \in I$ and $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ may be compactly open in $X \times Y$ and the set W_i may be compactly closed in $X \times Y$.

In the following, we assume that for each $i \in I$, Z_i is a topological vector space and $C_i : X \to 2^{Z_i}$ be such that for each $x \in X$, $C_i(x)$ is a closed convex cone with nonempty interior.

Definition 3.2 For each $i \in I$, $(x, y) \in X \times Y$, $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ is said to be

(i) $C_i(x)$-FC-quasiconvex in third argument if for each $(x, y) \in X \times Y$, $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >$, $\{z_{i,i_0}, \cdots z_{i,i_k}\} \subset N_i$, and $z_i \in \varphi_{N_i}(\Delta_k)$, there exists $j \in \{0, \cdots, n\}$ such that

$$\psi_i(x, y, z_{i,i_j}) \subset \psi_i(x, y, z_i) + C_i(x),$$

(ii) $C_i(x)$-FC-quasiconvex-like in third argument if for each $(x, y) \in X \times Y$, $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >$, $\{z_{i,i_0}, \cdots, z_{i,i_k}\} \subset N_i$ and $z_i \in \varphi_{N_i}(\Delta_k)$, there exists $j \in \{0, \cdots, k\}$ such that

$$\psi_i(x, y, z_i) \subset \psi_i(x, y, z_{i,i_j}) - C_i(x)$$

Remark 3.9 Definition 3.2 generalizes the corresponding notions of Lin [9] from convex subsets of topological vector spaces to FC-spaces.

Lemma 3.4 *If for each $i \in I$, $(x, y) \in X \times Y$, $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ is $C_i(x)$-FC-quasi-convex in third argument, then the sets*

$$\{z_i \in X_i : \psi_i(x, y, z_i) \not\subset C_i(x)\} \quad \text{and}$$

$$\{z_i \in X_i : \psi_i(x, y_i, z_i) \bigcap (-\text{int}C_i(x)) \neq \emptyset\}$$

are both FC-subspaces of X_i.

Proof If the set $\{z_i \in X_i : \psi_i(x, y, z_i) \not\subset C_i(x)\}$ is not a FC-subspace of X_i, then there exist $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >$, $\{z_{i,i_0}, \cdots, z_{i,i_k}\} \subset N_i \bigcap \{z_i \in X_i : \psi_i(x, y, z_i) \not\subset C_i(x)\}$ and $z_i^* \in \varphi_{N_i}(\Delta_k)$ such that $\psi_i(x, y, z_i^*) \subset C_i(x)$. Since $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ is $C_i(x)$-FC-quasiconvex in third argument, there exists $j \in \{0, \cdots, k\}$ such that

$$\psi_i(x, y, z_{i,i_j}) \subset \psi_i(x, y, z_i^*) + C_i(x) \subset C_i(x) + C_i(x) = C_i(x).$$

Since $z_{i,i_j} \in \{z_i \in X_i : \psi_i(x, y, z_i) \not\subset C_i(x)\}$, we have $\psi_i(x, y, z_{i,i_j}) \not\subset C_i(x)$ which is a contradiction. Hence the set $\{z_i \in X_i : \psi_i(x, y, z_i) \not\subset C_i(x)\}$ is a FC-subspace of X_i.
If the set $\{z_i \in X_i : \psi_i(x, y, z_i) \bigcap (-\text{int}C_i(x)) \neq \emptyset\}$ is not FC-subspace of X_i, then there

exist $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >, \{z_{i,i_0}, \cdots, z_{i,i_k}\} \subset N_i \bigcap \{z_i \in X_i : \psi_i(x, y, z_i) \bigcap (-\text{int}C_i(x)) \neq \emptyset\}$ and $z_i^* \in \varphi_{N_i}(\Delta_k)$ such that

$$\psi_i(x, y, z_i^*) \bigcap (-\text{int}C(x)) = \emptyset. \tag{3.3}$$

Since $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ is $C_i(x)$-FC-quasiconvex in third argument, there exists $j \in \{0, \cdots, k\}$ such that

$$\psi_i(x, y, z_{i,i_j}) \subset \psi_i(x, y, z_i^*) + C_i(x). \tag{3.4}$$

Since $z_{i,i_j} \in \{z_i \in X_i : \psi_i(x, y, z_i) \bigcap (-\text{int}C_i(x)) \neq \emptyset\}$, then we have

$$\psi_i(x, y, z_{i,i_j}) \bigcap (-\text{int}C_i(x)) \neq \emptyset. \tag{3.5}$$

Let $v_i^* \in \psi_i(x, y, z_{i,i_j}) \bigcap (-\text{int}C_i(x))$. By (3.4), there exists $u_i^* \in \psi_i(x, y, z_i^*)$ such that $v_i^* \in u_i^* + C_i(x)$. Hence we have that

$$u_i^* \in v_i^* - C_i(x) \subset -\text{int}C_i(x) - C_i(x) \subset -\text{int}C_i(x).$$

It follow that

$$\psi_i(x, y, z_i^*) \bigcap (-\text{int}C_i(x)) \neq \emptyset$$

which contradicts (3.3). Therefore the set $\{z_i \in X_i : \psi_i(x, y, z_i) \bigcap (-\text{int}C_i(x)) \neq \emptyset\}$ is FC-subspace of X_i.

Lemma 3.5 *If for each $i \in I$, $(x, y) \in X \times Y$, $\psi_i : X \times Y \times X_i \to 2^{Z_i}$ is $C_i(x)$-FC-quasi-convex-like in third argument, then the sets*

$$\{z_i \in X_i : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\} \quad \text{and}$$

$$\{z_i \in X_i : \psi_i(x, y, z_i) \bigcap C_i(x) = \emptyset\}$$

are both FC-subspaces of X_i.

Proof If the set $\{z_i \in X_i : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\}$ is not FC-subspace of X_i, then there exist $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >, \{z_{i,i_0}, \cdots, z_{i,i_k}\} \subset N_i \bigcap \{z_i \in X_i : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\}$, $z_i^* \in \varphi_{N_i}(\Delta_k)$ such that $\psi_i(x, y, z_i^*) \not\subset -\text{int}C_i(x)$. Since ψ_i is $C_i(x)$-FC-quasi-convex-like in third argument, there exists $j \in \{0, \cdots, k\}$ such that

$$\psi_i(x, y, z_i^*) \subset \psi_i(x, y, z_{i,i_j}) - C_i(x).$$

Since $z_{i,i_j} \in \{z_i \in X_i : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\}$, we have

$$\psi_i(x, y_i, z_{i,i_j}) \subset -\text{int}C_i(x).$$

It follows that $\psi_i(x, y, z_i^*) \subset \psi_i(x, y, z_{i,i_j}) - C_i(x) \subset -\text{int}C_i(x) - C_i(x) \subset -\text{int}C_i(x)$, which is a contradiction. Therefore the set $\{z_i \in X_i : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\}$ is a FC-subspace of X_i. If the set $\{z_i \in X_i : \psi_i(x, y, z_i) \bigcap C_i(x) = \emptyset\}$ is not FC-subspace of X_i, then there exist $N_i = \{z_{i,0}, \cdots, z_{i,n}\} \in < X_i >, \{z_{i,i_0}, \cdots, z_{i,i_k}\} \subset N_i \bigcap \{z_i \in X_i : \psi_i(x, y, z_i) \bigcap C_i(x) = \emptyset\}$, $z_i^* \in \varphi_{N_i}(\Delta_k)$ such that $\psi_i(x, y, z_i^*) \bigcap C_i(x) \neq \emptyset$. Let $u_i^* \in \psi_i(x, y, z_i^*) \bigcap C_i(x)$. Since ψ_i is $C_i(x)$-FC-quasiconvex-like in third argument, there exists $j \in \{0, \cdots, k\}$ such that

$$\psi_i(x, y, z_i^*) \subset \psi_i(x, y, z_{i,i_j}) - C_i(x).$$

Hence there exists $v_i^* \in \psi_i(x, y, z_{i,i_j})$ such that $u_i^* \in v_i^* - C_i(x)$ and so $v_i^* \in u_i^* + C_i(x) \subset C_i(x) + C_i(x) = C_i(x)$. It follows that $\psi_i(x, y, z_{i,i_j}) \bigcap C_i(x) \neq \emptyset$. Since $z_{i,i_j} \in \{z_i \in X_i : \psi_i(x, y_i, z_i) \bigcap C_i(x) = \emptyset\}$, we have $\psi_i(x, y, z_{i,i_j}) \bigcap C_i(x) = \emptyset$ which is a contradiction. Therefore the set $\{z_i \in X_i : \psi_i(x, y, z_i) \bigcap C_i(x) = \emptyset\}$ is a FC-subspace of X_i,

Theorem 3.5 *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I$, $C_i(x)$ is closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

- (i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $T_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, respectively,*
- (ii) *for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ are compactly open in $X \times Y$,*
- (iii) *the mapping $x \mapsto \text{int}C_i(x)$ has open graph and for each $z_i \in X_i$, the mapping $(x, y) \mapsto \psi_i(x, y, z_i)$ is upper semicontinuous on each compact subsets of $X \times Y$ with nonempty compact values,*
- (iv) *for each $y \in Y$, ψ_i is $C_i(x)$-FC-quasiconvex-like in third argument and for each $(x, y) \in X \times Y$, $\psi(x, y, x_i) \not\subset -\text{int}C_i(x)$,*
- (v) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in T_i(x, y)\}$ is compactly closed in $X \times Y$,*
- (vi) *for each $N_i \in < X_i >$, there exists compact FC-subspace L_{N_i} containing N_i and for each $M_i \in < Y_i >$, there exists compact FC-subspace L_{M_i} of Y_i containing M_i,*
- (vii) *for each $(x, y) \in X \times Y \setminus K \times H$, there exist $i \in I, \bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ and $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ such that $\psi_i(x, y, \bar{z}_i) \subset -\text{int}C_i(x)$.*

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \not\subset -\text{int}C_i(\hat{x}), \quad \forall\ z_i \in A_i(\hat{x}, \hat{y}).$$

Proof For each $i \in I$, define a set-valued mapping $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\}, \quad \forall\ (x, y) \in X \times Y.$$

By the condition (iii) and Lemma 3.3, for each $z_i \in Z_i$, the set $P_i^{-1}(z_i) = \{(x, y) \in X \times Y : \psi_i(x, y, z_i) \subset -\text{int}C_i(x)\}$ is compact open in $X \times Y$. It follows from (iv) and Lemma 3.5 that for each $i \in I$ and $(x, y) \in X \times Y$, $P_i(x, y)$ is a FC-subspaces of X_i and $x_i = \pi_i(x) \notin P_i(x, y)$. For each $i \in I$, define a set-valued mapping $G_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$G_i(x, y) = \begin{cases} [A_i(x, y) \bigcap P_i(x, y)] \times T_i(x, y), & \text{if } (x, y) \in W_i, \\ A_i(x, y) \times T_i(x, y), & \text{if } (x, y) \notin W_i, \end{cases}$$

By (i), for each $i \in I$ and $(x, y) \in X \times Y$, $G_i(x, y)$ is a FC-subspace of $X_i \times Y_i$. Since for each $i \in I$ and $(x, y) \in X \times Y$, $x_i \notin P_i(x, y)$, by the definition of W_i, we have that for each $i \in I$ and $(x, y) \in X \times Y$, $(x_i, y_i) \notin G_i(x, y)$. By using a similar argument as in the proof of Theorem 3.1, we can show that for each $i \in I$ and $(u_i, v_i) \in X_i \times Y_i$, $G_i^{-1}(u_i, v_i)$ is compactly open in $X \times Y$. By (vi), for each $H_i = N_i \times M_i \in < X_i \times Y_i >$ there exists compact FC-subspace $L_{H_i} = L_{N_i} \times L_{M_i}$ of $X_i \times Y_i i$ containing H_i. By (vii), for each $(x, y) \in X \times Y \setminus K \times H$, there exists $i \in I$ such that $G_i(x, y) \bigcap L_{H_i} \neq \emptyset$. All conditions of Lemma 2.4 are satisfied. The rest of the proof is similar to that of Theorem 3.1. We omit.

Theorem 3.6 *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I$, $C_i(x)$ is closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

- (i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $T_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, respectively,*

(ii) *for each* $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ *are compactly open in* $X \times Y$,
(iii) *the mapping* $x \mapsto C_i(x)$ *is upper semicontinuous on* X *and for each* $z_i \in Z_i$, *the mapping* $(x, y) \mapsto \psi_i(x, y, z_i)$ *is lower semicontinuous on each compact subset of* $X \times Y$,
(iv) *for each* $y \in Y$, ψ_i *is* $C_i(x)$*-FC-quasiconvex in third argument and for each* $(x, y) \in X \times Y$, $\psi(x, y, x_i) \subset C_i(x)$,
(v) *the set* $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in T_i(x, y)\}$ *is compactly closed in* $X \times Y$,
(vi) *for each* $N_i \in < X_i >$, *there exists compact* FC*-subspace* L_{N_i} *containing* N_i *and for each* $M_i \in < Y_i >$, *there exists compact* FC*-subspace* L_{M_i} *of* Y_i *containing* M_i,
(vii) *for each* $(x, y) \in X \times Y \setminus K \times H$, *there exist* $i \in I, \bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ *and* $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ *such that* $\psi_i(x, y, \bar{z}_i) \not\subset C_i(x)$.

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \subset C_i(\hat{x}), \quad \forall\ z_i \in A_i(\hat{x}, \hat{y})$$

i.e., $(\hat{x}, \hat{y})$ *is a solution of the SGVQEP(II).*

Proof For each $i \in I$, define a set-valued mapping $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : \psi_i(x, y, z_i) \not\subset C_i(x)\}, \quad \forall\ (x, y) \in X \times Y.$$

By the condition (iii) and Lemma 3.2, for each $z_i \in Z_i$. $P^{-1}(z_i) = \{(x, y) \in X \times Y : \psi_i(x, y, z_i) \not\subset C_i(x)\}$ is compact open in $X \times Y$. It follows from (iv) and Lemma 3.4 that for each $i \in I$ and $(x, y) \in X \times Y$, $P_i(x, y)$ is a FC-subspaces of X_i and $x_i \notin P_i(x, y)$. The rest of the proof is similar to that of Theorem 3.5. we omit it.

By applying Lemmas 2,4, 3.2–3.4 and the similar argument as in the proof of Theorems 3.5 and 3.6, we can easily prove the following results.

Theorem 3.7 *Let* K *and* H *be nonempty compact subsets of* X *and* Y, *respectively, and for each* $i \in I$, $C_i(x)$ *is closed convex cone with nonempty interior. Suppose that for each* $i \in I$,

(i) *for each* $(x, y) \in X \times Y$, $A_i(x, y)$ *and* $T_i(x, y)$ *are both nonempty* FC*-subspaces of* X_i *and* Y_i, *respectively,*
(ii) *for each* $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ *are compactly open in* $X \times Y$,
(iii) *the mapping* $x \mapsto \text{int}C_i(x)$ *has open graph and for each* $z_i \in X_i$, *the mapping* $(x, y) \mapsto \psi_i(x, y, z_i)$ *is lower semicontinuous on each compact subsets of* $X \times Y$,
(iv) *for each* $y \in Y$, ψ_i *is* $C_i(x)$*-*FC*-quasiconvex in third argument and for each* $(x, y) \in X \times Y$, $\psi(x, y, x_i) \bigcap (-\text{int}C_i(x)) = \emptyset$,
(v) *the set* $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in T_i(x, y)\}$ *is compactly closed in* $X \times Y$,
(vi) *for each* $N_i \in < X_i >$, *there exists compact* FC*-subspace* L_{N_i} *containing* N_i *and for each* $M_i \in < Y_i >$, *there exists compact* FC*-subspace* L_{M_i} *of* Y_i *containing* M_i,
(vii) *for each* $(x, y) \in X \times Y \setminus K \times H$, *there exist* $i \in I$, $\bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ *and* $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ *such that* $\psi_i(x, y, \bar{z}_i) \bigcap (-\text{int}C_i(x)) \neq \emptyset$.

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \bigcap (-\text{int}C_i(\hat{x})) = \emptyset, \quad \forall\ z_i \in A_i(\hat{x}, \hat{y}).$$

Theorem 3.8 *Let* K *and* H *be nonempty compact subsets of* X *and* Y, *respectively, and for each* $i \in I$, $C_i(x)$ *is closed convex cone with nonempty interior. Suppose that for each* $i \in I$,

(i) *for each* $(x, y) \in X \times Y$, $A_i(x, y)$ *and* $T_i(x, y)$ *are both nonempty* FC*-subspaces of* X_i *and* Y_i*, respectively,*
(ii) *for each* $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $T_i^{-1}(v_i)$ *are compactly open in* $X \times Y$*,*
(iii) *the mapping* $x \mapsto C_i(x)$ *is upper semicontinuous on* X *and for each* $z_i \in Z_i$*, the mapping* $(x, y) \mapsto \psi_i(x, y, z_i)$ *is upper semicontinuous on each compact subset of* $X \times Y$ *with nonempty compact values,*
(iv) *for each* $y \in Y$, ψ_i *is* $C_i(x)$*-FC-quasiconvex-like in third argument and for each* $(x, y) \in X \times Y$, $\psi(x, y, x_i) \bigcap C_i(x) \neq \emptyset$*,*
(v) *the set* $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in T_i(x, y)\}$ *is compactly closed in* $X \times Y$*,*
(vi) *for each* $N_i \in <X_i>$*, there exists compact* FC*-subspace* L_{N_i} *containing* N_i *and for each* $M_i \in <Y_i>$*, there exists compact* FC*-subspace* L_{M_i} *of* Y_i *containing* M_i*,*
(vii) *for each* $(x, y) \in X \times Y \setminus K \times H$*, there exist* $i \in I$, $\bar{z}_i \in A_i(x, y) \bigcap L_{N_i}$ *and* $\bar{y}_i \in T_i(x, y) \bigcap L_{M_i}$ *such that* $\psi_i(x, y, \bar{z}_i) \bigcap C_i(x) = \emptyset$.

Then there exists $(\hat{x}, \hat{y}) \in K \times H$ *such that for each* $i \in I$*,*

$$\hat{x}_i \in A_i(\hat{x}, \hat{y}), \quad \hat{y}_i \in T_i(\hat{x}, \hat{y}) \quad \text{and} \quad \psi_i(\hat{x}, \hat{y}, z_i) \bigcap C_i(\hat{x}) \neq \emptyset, \quad \forall\ z_i \in A_i(\hat{x}, \hat{y})$$

i.e., $(\hat{x}, \hat{y})$ *is a solution of the SGVQEP(IV).*

Remark 3.10 Theorems 3.5–3.8 also generalize the corresponding results of Peng et al. [5] from convex subsets of topological spaces to FC-spaces under much weak assumptions.

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